# Sampling-Based Estimation of DQPSK Transmission Bit Error Rates with Nakagami-*m* Fading Channels

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***Abstract***

A general Monte Carlo estimator for calculation of bit error rates for a class of noncoherent and differentially coherent communication systems is introduced. This estimator, by exploiting the connection between the Generalised Marcum Q-Function and Poisson distributions, is applied to the case of differential quadrature phased-shift keying over a frequency-selective Nakagami-*m* channel with an arbitrary power delay profile. It is confirmed that this estimator is efficient, and provides good results in practice.

***Keywords***

*Communication System Performance; Bit Error Rates; Fading Channels; Estimation; Monte Carlo Methods*

## Bit Error Rates and Marcum's Q-Function

Bit error rates (BERs) are an important performance measure for communication systems, and consequently, their estimation is of considerable interest (Proakis 2001, Simon and Alouini 2000*a*).

This paper examines the Monte Carlo estimation of BERs for a class of noncoherent and differentially coherent communications systems, operated over frequency selective channels and *L*-fold diversity reception (Simon and Alouini 2000*b*). These BERs depend on the well-known Marcum Q-Function (Marcum 1960), which provides an estimation challenge. It has proven useful to apply bounds on the Marcum Q-Function to approximate these BERs, enabling the determination of minimum and maximum performance levels (Simon and Alouini 2000*b*, Chiani 1999, Ferrari and Corazza 2004). Recently, it has been shown that the Marcum Q-Function can be related to discrete Poisson distributions (Weinberg 2006). This connection facilitates the sampling-based estimation of the Marcum Q-Function. Consequently, Monte Carlo estimators, using a Poisson sampling distribution, can be constructed to estimate it (see Weinberg 2006 for a series of such estimators). Additionally, it has also been demonstrated that the work of (Weinberg 2006) can be applied to BER estimation. In (Weinberg 2008), a Monte Carlo estimator is produced for a simple case of differential quadrature phased-shift keying (DQPSK). It is shown that this can be extended to provide a Monte Carlo solution to a more general BER estimation problem.

Monte Carlo methods utilise the statistical Strong Law of Large Numbers (SLLN), which permits the estimation of expectations through simulation of random numbers (Robert and Casella 2004). The SLLN guarantees that a Monte Carlo estimator will converge, but the issue is whether an accurate approximation is obtained within a reasonable sample size. This is the tradeoff in using a very simple scheme to approximate an intractable result.

The BER under consideration is given by

$$\mathrm{BER}(\Xi_t) = \frac{1}{2} + \frac{1}{2^{2L-1}} \sum_{n=1}^{L} \binom{2L-1}{L-n} \times \left[ Q_n(a\sqrt{\Xi_t}, b\sqrt{\Xi_t}) - Q_n(b\sqrt{\Xi_t}, a\sqrt{\Xi_t}) \right], \quad (1)$$

where constants $a$ and $b$ are given by

$$a = \sqrt{2-\sqrt{2}}, \quad b = \sqrt{2+\sqrt{2}},$$

and $Q_n$ is the generalized Marcum Q-Function of order $n$, given by

$$Q_n(\alpha, \beta) = \frac{1}{\alpha^{n-1}} \int_{\beta}^{\infty} x^n e^{-\left(\frac{x^2+\alpha^2}{2}\right)} I_{n-1}(\alpha x) dx,$$

$I_n(x)$ is the modified Bessel function of order $n$, and

$$\Xi_t = \sum_{l=1}^{L} \Xi_l$$

is the total instantaneous signal to noise ratio (TISNR) per bit of all $L$ paths (see Proakis 2001). Under the Nakagami $m$-channel model, with $L = 1$, the TISNR is modelled as a Gamma random variable with shape parameter $m$ and mean the average received signal to noise ratio. Similarly, the case where $L$=1, and the TISNR is a deterministic constant, has been addressed in a number of papers, including (Ferrari and Corazza 2004 and Weinberg 2008). This corresponds to DQPSK with Gray coding over an additive white noise channel (AWGN).

In (Ferrari and Corazza 2004) bounds are used to estimate the BER, while in (Weinberg 2008) a Monte Carlo estimator is used. In the case where TISNR is random, such as in DQPSK over a frequency selective Nakagami $m$-channel, with an arbitrary power delay profile, the situation is more complex. In these scenarios, TISNR is a sum of random variables, and we are interested in the mean of (1).

Bounds on the BER in this case have been used extensively to determine performance levels, as in (Simon and Alouini 2000*b* and Chiani 1999).

In (Weinberg 2008) a simple Monte Carlo estimator is used to estimate (1) for the case of $L$=1. However, this estimator is inefficient. This is because it requires sampling from a correlated pair of random variables, and uses no compression of data through conditioning (Bucklew 2005).

In particular, it is shown in (Weinberg 2008) that the BER for this case can be written as a function of two correlated negative binomial distributions, and the sampling is performed through simulating this pair. This can be done using an algorithm based upon Poisson-stopped sums of independent logarithmic series random variables (Johnson, Kemp and Kotz 2005). However, the performance of such algorithms can be affected by the correlation, and this negatively affected the behaviour of the estimator in (Weinberg 2008).

It is shown how this situation can be improved using conditional distributions. Specifically, we will be interested in the application of the approach of (Bucklew 2005). A general purpose Monte Carlo estimator of (1) will be proposed, and it will be analysed through a specific example.

The paper is organised as follows. Section 2 introduces the general estimator. Section 3 provides the mathematical tools to apply this estimator in a simple scenario. Section 4 analyses the performance of this estimator, in the scenario of Section 3 through simulations.

Throughout the following we use **P** to denote probability, **E** to be statistical mean with respect to **P**, **V** is statistical variance, **Cov** is covariance, $\mathbf{N} = \{0,1,2,\ldots\}$ and $\mathbf{N}^+ = \mathbf{N} - \{0\}$. The indicator set function is $\mathbf{I}[x \in A] = 1$ if and only if $x \in A$ and is zero otherwise.

## General Estimator

The construction of a general Monte Carlo estimator of (1) requires the application of the Poisson-Marcum Q-Function association derived from (Weinberg 2006).

The key result in the latter is that if $X_1(\mu_1)$ and $X_2(\mu_2)$ are a pair of independent Poisson random variables with means $\mu_1$ and $\mu_2$ respectively, then (2) can be written as

$$\mathrm{Q}_n(\alpha,\beta) = \mathbf{P}\left(X_1\left(\frac{\beta^2}{2}\right) \le n-1+X_2\left(\frac{\alpha^2}{2}\right)\right). \tag{3}$$

Without loss of generality, these Poisson random variables are assumed to be independent of TISNR by construction. This allows us to work directly with Poisson distributions with randomised means, free from writing down expectations with respect to the TISNR in the following.

Equation (3) can be applied to (1) to construct a Monte Carlo estimator. For brevity, two random variables are defined as

$$Z_1 = X_1\left(\frac{b^2\Xi_t}{2}\right) \text{ and } Z_2 = X_2\left(\frac{a^2\Xi_t}{2}\right).$$

Note that

$$\begin{aligned}&\mathrm{Q}_n(a\sqrt{\Xi_t}, b\sqrt{\Xi_t}) - \mathrm{Q}_n(b\sqrt{\Xi_t}, a\sqrt{\Xi_t})\\ =\ &\mathbf{P}(Z_1 \le n-1+Z_2) - \mathbf{P}(Z_2 \le n-1+Z_1),\end{aligned} \tag{4}$$

where in the second probability in (4), we have switched the two Poisson random variables without loss of generality. By conditioning on $Z_2$, and allowing for dependence between $Z_1$ and $Z_2$, we have

$$\begin{aligned}&\mathrm{Q}_n(a\sqrt{\Xi_t}, b\sqrt{\Xi_t}) - \mathrm{Q}_n(b\sqrt{\Xi_t}, a\sqrt{\Xi_t})\\ =\ &\sum_{j=0}^{\infty}\mathbb{P}(Z_2 = j)\times\\ &[\mathbb{P}(Z_1 \le n-1+j|Z_2=j) - \mathbb{P}(Z_1 \ge j-n+1|Z_2=j)].\end{aligned} \tag{5}$$

For a fixed $n \in \mathbf{N}^+$, a function $g_n$: $\mathbf{N} \dashrightarrow [0, 2]$ is defined by

$$g_n(j) = \mathbf{P}(Z_1 \le j+n-1|Z_2=j] + \mathbf{P}(Z_1 \le j-n+1|Z_2=j). \tag{6}$$

This is a sum of cumulative distribution functions of $Z_1$ conditioned on $Z_2$. Then by application of (6) to (5), and substitution of the result into (1), the mean bit error rate is

$$\mathbf{E}[\mathrm{BER}(\Xi_t)] = 2^{-1} - 2^{-2L+1}\sum_{n=1}^{L}\binom{2L-1}{L-n}\sum_{j=0}^{\infty}\mathbf{P}(Z_2=j)(1-g_n(j)). \quad (7)$$

Note that since TISNR is considered to be random its mean value is of interest. The expression (7) is in a useful form from a Monte Carlo simulation point of view (Robert and Casella 2004). Additionally it also provides an interesting probabilistic form of the mean BER. In particular, it is observed that

$$\sum_{j=0}^{\infty}\mathbf{P}(Z_2=j)(1-g_n(j)) = 1-\mathbf{E}[g_n(Z_2)].$$

Consequently, (7) is the mean of a function of a Poisson random variable with a randomised mean value. It is remarked that this may provide a means of constructing bounds on (1). This will be examined in future research directions.

In view of (7), a Monte Carlo estimator is defined

$$\hat{P} = 2^{-1} - 2^{-2L+1}\sum_{n=1}^{L}\binom{2L-1}{L-n}\frac{1}{N}\sum_{k=1}^{N}(1-g_n(Y_{k,n})), \quad (8)$$

where $\{Y_{k,n}, k \in \{1,2,\ldots,N\}, n \in \{1,2,\ldots,L\}\}$ consist of independent and identically distributed copies of $Z_2$.

Note that it is mathematically convenient to ensure that these sampling random variables are independent over all pairs of indicies $(k,n)$. This is an unbiased estimator of (1), which can be shown by taking the mean of (8), noting that each $Y_{k,n}$ has the same distribution as $Z_2$, and using (4). By means of the fact that variance is additive over independent random variables, and $\mathbf{V}(aX) = a^2\mathbf{V}(X)$ for constant $a$, the variance of (8) is

$$\mathbf{V}(\hat{P}) = 2^{-4L+2}\sum_{n=1}^{L}\binom{2L-1}{L-n}^2\frac{\mathbf{V}(g_n(Z_2))}{N}. \quad (9)$$

Further expansion of (9), through evaluation of the variance of $g_n$ under $Z_2$, does not provide insights into the performance of (8) in general. It is observed that (9) is of order $1/N$, but in practical applications, it may be possible that the variance is of smaller order. However, it is expected that (8) is much more efficient than the corresponding estimator in (Weinberg 2008). This is due to the compression of data through conditioning (Bucklew 2005). This claim will be investigated in Section 4 for a particular example.

Additionally, it is noted that Importance Sampling could be used to improve the convergence of (8). However, in view of the analysis in (Weinberg 2006), it would be expected that only minor improvements are possible.

In the practical implementation of (8), the knowledge of the distribution of $Z_2$ and the cumulative distribution function of $Z_1 \mid Z_2$ are required. In the next section we derive these in the case of DPQSK under Nakagami Fading.

## Application to DQPSK under Nakagami Fading

To illustrate the application of (8) DQPSK over a frequency selective Nakagami $m$-channel model is examined, as described in (Simon and Alouini 2000*b*). All the necessary tools are derived to apply (8) to estimate BERs in this context. For simplicity, our attention will be restricted to the case where $L$=1. In this scenario, the estimator (8) becomes

$$\hat{P} = \frac{1}{2N}\sum_{k=1}^{N} g_1(Y_k), \quad (10)$$

where each $Y_k := Y_{k,1}$ has the same distribution as $Z_2$. Under the Nakagami $m$-channel model, the TISNR has a Gamma distribution with density

$$f_{\Xi_t}(\xi) = \frac{m^m}{\Gamma(m)\gamma_{av}{}^m}\xi^{m-1}e^{-\frac{m\xi}{\gamma_{av}}},$$

where $\gamma_{av}$ is the average received signal to noise ratio and $\Gamma$ is the Gamma function (Simon and Alouini 2000*b*). Since $Z_1$ and $Z_2$ are Poisson random variables with means a constant times a Gamma variable, we can condition on this common distribution to extract the necessary distributions for (10). In particular, $(Z_1, Z_2)$ can be shown to have jointly dependent Negative Binomial marginal distributions (see Johnson, Kemp and Kotz 2005), with covariance matrix

$$\mathcal{C}(Z_1,Z_2) = \begin{bmatrix} \gamma_{av}\nu_1 + \nu_1^2\frac{\gamma_{av}{}^2}{m} & \frac{\gamma_{av}{}^2}{2m} \\ \frac{\gamma_{av}{}^2}{2m} & \gamma_{av}\nu_2 + \nu_2^2\frac{\gamma_{av}{}^2}{m} \end{bmatrix}, \quad (11)$$

where $\mu_1 = 0.5b^2$ and $\mu_2 = 0.5a^2$. Note that in matrix (11), the entry $a_{ij} = \mathbf{Cov}(Z_i, Z_j) = \mathbf{E}(Z_iZ_j) - \mathbf{E}(Z_i)\mathrm{E}(Z_j)$ for all $i$, j $\varepsilon\{1, 2\}$. The main diagonal entries in (11) are calculated using the variances of the marginal distributions. The term $a_{12} = a_{21}$ is obtained based the fact that the random variables

$$Z_i|\{\Xi_t = \xi\}$$

are independent Poissons with mean $\upsilon_i\xi$ and

$$\mathbf{E}(Z_1Z_2) = \mathbf{E}(\mathbf{E}[Z_1Z_2|\Xi_t]) = \mathbf{E}(\mathbf{E}[Z_1|\Xi_t]\mathbf{E}[Z_2|\Xi_t])$$

and then application of the corresponding moment expressions for a Gamma distribution. It is observed that since $a_{12} \geq 0$ in (11), the marginal random variables $Z_1$ and $Z_2$ are positively correlated.

The marginal distribution $Z_2$ can be shown to be Negative Binomial with parameters $m$ and

$$\theta = \frac{m}{\gamma_{av}(\nu_2 + \frac{m}{\gamma_{av}})}$$

and point probabilities given by

$$\mathbf{P}(Z_2 = k) = \frac{\Gamma(m+k)}{\Gamma(m)\Gamma(k+1)}\theta^m(1-\theta)^k, \qquad (12)$$

for $k \in \mathbf{N}$ (see Johnson, Kemp and Kotz 2005). This is our sampling distribution. By outlining how to construct the conditional distribution of $Z_1 \mid Z_2$, we begin to construct the joint probability mass function of $Z_1$ and $Z_2$.

It is recalled that when conditioned on the TISNR, these variables become independent Poissons. Hence for any appropriate indicies $i$ and $j$,

$$\begin{aligned}
\mathbf{P}(Z_1 = i, Z_2 = j) \\
&= \int_0^\infty \mathbf{P}(\aleph_1(\nu_1\xi) = i, \aleph_2(\nu_2\xi) = j) f_{\Xi_t(\xi)} d\xi \\
&= \frac{\nu_1^i \nu_2^j m^m}{i!j!\Gamma(m){\gamma_{av}}^m} \int_0^\infty \xi^{m+i+j-1} e^{-(\nu_1+\nu_2+\frac{m}{\gamma_{av}})} d\xi \\
&= \frac{\Gamma(m+i+j)}{i!j!\Gamma(m)} \frac{m^m}{{\gamma_{av}}^m} \frac{\nu_1^i \nu_2^j}{\left(2+\frac{m}{\gamma_{av}}\right)^{(m+i+j)}}. \qquad (13)
\end{aligned}$$

Consequently, by dividing (13) into (12), we arrive at the conditional distribution

$$\begin{aligned}
\mathbf{P}(Z_1 = i | Z_2 = j) &= \frac{\mathbf{P}(Z_1 = i, Z_2 = j)}{\mathbf{P}(Z_2 = j)} \\
&= \binom{m+j-1+i}{i} \left(\frac{\nu_1}{2+\frac{m}{\gamma_{av}}}\right)^i \left(\frac{\nu_2+\frac{m}{\gamma_{av}}}{2+\frac{m}{\gamma_{av}}}\right)^{m+j}. (14)
\end{aligned}$$

This implies the conditional distribution is a power series distribution (Johnson, Kemp and Kotz 2005). The estimator (10) requires us to sample (12) and evaluate cumulative distribution functions of (14) through the function (6) with $n$=1. This can be implemented in a simple numerical algorithm. The following Section provides an analysis of (10) in the current context.

## Numerical Analysis

We now examine the performance of (10) for the example outlined in the previous Section. This analysis will be twofold. Firstly, we compare (10) to the corresponding estimator proposed in (Weinberg 2008) and will explain why the estimator (10) has better performance. Secondly, we will compare simulations of (10) to results obtained via numerical integration.

### Estimator Comparison

The estimator proposed in (Weinberg 2008) is

$$\hat{Q} = \frac{1}{N}\sum_{k=1}^{N} h(X_1^k, X_2^k), \qquad (15)$$

where $h(i, j) = \mathbf{I}[i \leq j] - 0.5\mathbf{I}[i = j]$, and for each $k$, $(X_1^k, X_2^k)$ has the same joint distribution as $(Z_1, Z_2)$. This estimator requires sampling from the joint distribution (13) directly. In order to simulate the bivariate pair ($Z_1$, $Z_2$), an overlapping sums (OS) algorithm is in use developed by Oregon State University. This algorithm's performance is heavily dependent on the correlation structure (11). In particular, it was found that in cases of interest, the algorithm could not produce simulations, and consequently, no Monte Carlo estimate could be obtained. This tended to occur as $\gamma_{av}$ increased. For smaller $\gamma_{av}$, and especially with larger $m$, the joint distribution is decoupled, due to the fact that (11) becomes approximately diagonal. Here, the algorithm works well. However, in general, cases where the algorithm could be applied take a significant number of simulations ($N >> 10^6$) to produce results consistent with bounds on the BER. The reason for the failure of the OS algorithm is due to the fact that it cannot tolerate cases where the means and variances of the marginal distributions are significantly different (Madsen and Dalthorp 2005).

A comparison is made between the variances of (10) and (15). Using the definition of variance, it can be shown that

$$\mathbf{V}(\hat{P}) = \frac{1}{N}\left[0.25\mathbf{E}[g_1^2(Z_2)] - \mathbf{E}^2[\mathrm{BER}(\Xi_t)]\right], \qquad (16)$$

where

$$\mathbf{E}[\mathrm{BER}(\Xi_t)] = \mathbf{P}(Z_1 \leq Z_2) - 0.5\mathbf{P}(Z_1 = Z_2)$$

(since it is an unbiased estimator of this average BER) and

$$\begin{aligned}
\mathbf{E}[g_1^2(Z_2)] &= \sum_{k=0}^{\infty} \mathbf{P}(Z_2 = k)\Big(\mathbf{P}(Z_1 \leq k | Z_2 = k)^2 \\
&\quad + 2\mathbf{P}(Z_1 \leq k | Z_2 = k)\mathbf{P}(Z_1 \leq k-1 | Z_2 = k) \\
&\quad + \mathbf{P}(Z_1 \leq k-1 | Z_2 = k)^2\Big). \qquad (17)
\end{aligned}$$

By a straightforward calculation, it can be shown that

$$\mathbf{V}(\widehat{Q}) = \frac{1}{N}\left[\mathbf{P}(Z_1 \leq Z_2) - 0.75\mathbf{P}(Z_1 = Z_2) - \mathbf{E}^2[\mathrm{BER}(\Xi_t)]\right]. \quad (18)$$

Since probabilities are smaller than 1, the sum of three terms inside the brackets in (17) is bounded above by

$$\mathbf{P}(Z_1 \leq k | Z_2 = k) + 3\mathbf{P}(Z_1 \leq k-1 | Z_2 = k),$$

and consequently by application of this bound to (17) and (16), it can be shown that

$$\mathbf{V}(\widehat{P}) << \mathbf{V}(\widehat{Q}),$$

implying (10) is a more efficient estimator than (15).

Hence, from a mathematical point of view, it is clear that (10) is a better option for Monte Carlo estimation of BERs. For this reason, the estimator (15) will be considered redundant in the numerical analysis to follow.

### *Simulations*

We now examine the performance of the estimator (10) through simulation. The case of $m$ ε {1, 3.5, 5, 10} is considered. These estimates are compared to a corresponding result based upon adaptive Simpson Quadrature. In particular, the algorithmic results in (Tanda 1993) have been used for this purpose. Throughout, the absolute value of the relative error is used, and defined as |ASQ-MC|/ASQ, where ASQ is the numerical estimate from (Tanda 1993) and MC is the Monte Carlo estimate produced through the estimator (10).

The first case considered is that $m$=1. Table 1 shows the result of applying a Monte Carlo estimator with a sample size of $N$=50,000, while Table 2 shows the effect of increasing $N$ to 100,000. An interesting observation is that the relative errors are roughly of the same order for the two sample sizes, which is indicative of a slow rate of convergence of (10). Likewise, also it is noted that the errors are fairly small for the first half of $\gamma_{av}$ values, then they tend to increase. Both tables show an estimate of the variance of (10), which is of the same order for both tables.

TABLE 1 $m$=1, $N$ = 50,000

| $\gamma_{av}$ (dB) | Quadrature | Monte Carlo | Relative Error | Estimator Variance |
|---|---|---|---|---|
| 0 | 2.32738644374792e-001 | 2.32794935367333e-001 | 2.4186e-004 | 0.1582e-007 |
| 1 | 2.06598981168838e-001 | 2.06602152445031e-001 | 1.5349e-005 | 0.0852e-007 |
| 2 | 1.81275217005844e-001 | 1.81287142760762e-001 | 6.5788e-005 | 0.0146e-007 |
| 3 | 1.57242162122411e-001 | 1.57296987463936e-001 | 3.4866e-004 | 0.0434e-007 |
| 4 | 1.34896307063021e-001 | 1.34939526308406e-001 | 3.2038e-004 | 0.0725e-007 |
| 5 | 1.14523097597356e-001 | 1.14543005091449e-001 | 1.7382e-004 | 0.1533e-007 |
| 6 | 9.62914735631688e-002 | 9.62529876258401e-002 | 3.9968e-004 | 0.1957e-007 |
| 7 | 8.02552070717308e-002 | 8.02582040375331e-002 | 3.7343e-005 | 0.2297e-007 |
| 8 | 6.63696658688111e-002 | 6.65804273348345e-002 | 3.1756e-003 | 0.2513e-007 |
| 9 | 5.45131573969273e-002 | 5.45096927481960e-002 | 6.3556e-005 | 0.2588e-007 |
| 10 | 4.45124042448613e-002 | 4.45562556209769e-002 | 9.8515e-004 | 0.2372e-007 |
| 11 | 3.61653579195491e-002 | 3.63870482881961e-002 | 6.1299e-003 | 0.2191e-007 |
| 12 | 2.92606626380140e-002 | 2.94873337364886e-002 | 7.7466e-003 | 0.2014e-007 |
| 13 | 2.35919261485608e-002 | 2.37065743732054e-002 | 4.8596e-003 | 0.1814e-007 |
| 14 | 1.89668889405030e-002 | 1.90044134126462e-002 | 1.9784e-003 | 0.1521e-007 |
| 15 | 1.52127767079031e-002 | 1.51960983685304e-002 | 1.0963e-003 | 0.1308e-007 |
| 16 | 1.21784190218376e-002 | 1.22450817102033e-002 | 5.4738e-003 | 0.1084e-007 |
| 17 | 9.73420722930446e-003 | 9.91650562970578e-003 | 1.8728e-002 | 0.0904e-007 |
| 18 | 7.77084581782377e-003 | 7.81333814350508e-003 | 5.4682e-003 | 0.0698e-007 |
| 19 | 6.19726134707945e-003 | 6.24224887686460e-003 | 7.2592e-003 | 0.0570e-007 |
| 20 | 4.93821812087230e-003 | 5.33725222009707e-003 | 8.0805e-002 | 0.0459e-007 |

TABLE 2 $m$=1, $N$=100,000

| $\gamma_{av}$ (dB) | Quadrature | Monte Carlo | Relative Error | Estimator Variance |
|---|---|---|---|---|
| 0 | 2.32738644374792e-001 | 2.32763638395201e-001 | 1.0739e-004 | 0.0896e-007 |
| 1 | 2.06598981168838e-001 | 2.06516281743548e-001 | 4.0029e-004 | 0.0437e-007 |
| 2 | 1.81275217005844e-001 | 1.81146295849834e-001 | 7.1119e-004 | 0.0099e-007 |
| 3 | 1.57242162122411e-001 | 1.57294755117851e-001 | 3.3447e-004 | 0.0174e-007 |
| 4 | 1.34896307063021e-001 | 1.34893387992866e-001 | 2.1639e-005 | 0.0367e-007 |
| 5 | 1.14523097597356e-001 | 1.14534728540521e-001 | 1.0156e-004 | 0.0724e-007 |
| 6 | 9.62914735631688e-002 | 9.64395745936454e-002 | 1.5380e-003 | 0.0979e-007 |
| 7 | 8.02552070717308e-002 | 8.02836021493042e-002 | 3.5381e-004 | 0.1222e-007 |
| 8 | 6.63696658688111e-002 | 6.63402632756448e-002 | 4.4301e-004 | 0.1247e-007 |
| 9 | 5.45131573969273e-002 | 5.45678089036584e-002 | 1.0025e-003 | 0.1326e-007 |
| 10 | 4.45124042448613e-002 | 4.44208327718689e-002 | 2.0572e-003 | 0.1250e-007 |
| 11 | 3.61653579195491e-002 | 3.63109154280842e-002 | 4.0248e-003 | 0.1126e-007 |
| 12 | 2.92606626380140e-002 | 2.93158150931747e-002 | 1.8849e-003 | 0.1015e-007 |
| 13 | 2.35919261485608e-002 | 2.36798907105401e-002 | 3.7286e-003 | 0.0918e-007 |
| 14 | 1.89668889405030e-002 | 1.88536675715670e-002 | 5.9694e-003 | 0.0753e-007 |
| 15 | 1.52127767079031e-002 | 1.52329338185743e-002 | 1.3250e-003 | 0.0632e-007 |
| 16 | 1.21784190218376e-002 | 1.22856748220960e-002 | 8.8070e-003 | 0.0532e-007 |
| 17 | 9.73420722930446e-003 | 9.66646624362696e-003 | 6.9590e-003 | 0.0440e-007 |
| 18 | 7.77084581782377e-003 | 7.87330134660896e-003 | 1.3185e-002 | 0.0339e-007 |
| 19 | 6.19726134707945e-003 | 6.34254524941773e-003 | 2.3443e-002 | 0.0297e-007 |
| 20 | 4.93821812087230e-003 | 5.22358355594688e-003 | 5.7787e-002 | 0.0242e-007 |

Figure 1 is a plot of the logarithmic BER as a function of $\gamma_{av}$ (in dB), for increasing sample sizes $N$.

In this case we have set $m$=1. What can be observed from Figure 1 is the fast convergence of the estimator (10). Figure 2 shows a magnification of Figure 1 to further illustrate this, which indicates that in practice the estimator (10) will not require a significantly large, sample size, and hence excluding from a heavy computational load.

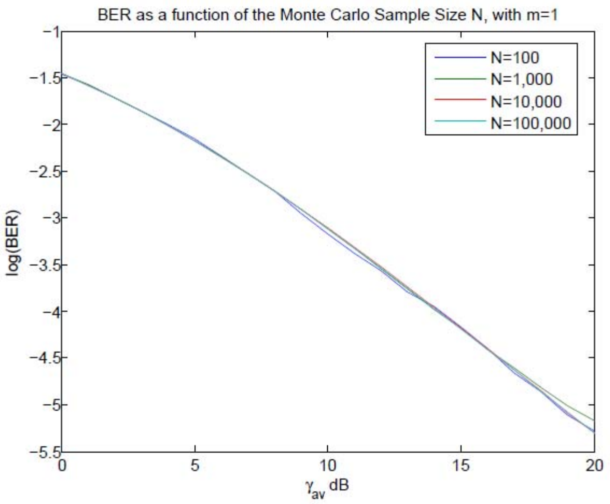


FIG. 1 AN EXAMPLE OF LOGARITHMIC BER FOR VARYING SAMPLE SIZES

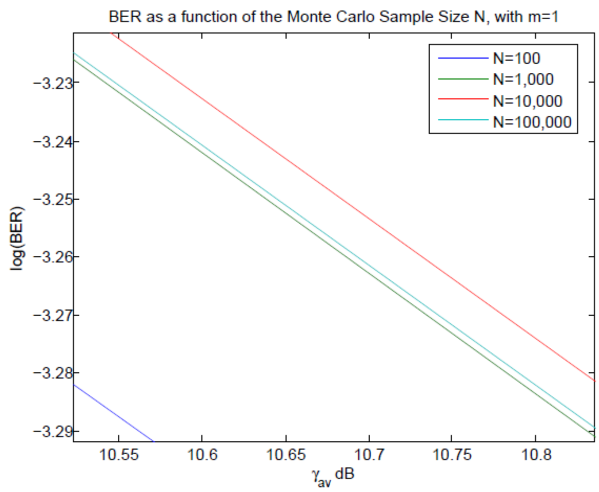


FIG. 2 MAGNIFICATION OF THE PLOTS IN FIGURE 1

Next, an example of nonintegral $m$ is in consideration. Table 3 contains Monte Carlo simulations with the same sample size of $N$=50,000. We see behaviour similar to the case of $m$=1, except that the results tend to be much better, in terms of maintaining a smaller relative error. It is interesting to note that the Monte Carlo variance becomes very small as $\gamma_{av}$ increases. The table shows estimates only up to $\gamma_{av}$ = 17 dB because of difficulty in computing an accurate variance for larger $\gamma_{av}$.

TABLE 3 $m$=3.5, $N$=100,000

| $\gamma_{av}$ (dB) | Quadrature | Monte Carlo | Relative Error | Estimator Variance |
|---|---|---|---|---|
| 0 | 1.87583912502049e-001 | 1.87193511998927e-001 | 2.0812e-003 | 0.9056e-006 |
| 1 | 1.56094004701240e-001 | 1.56027644117283e-001 | 4.2513e-004 | 0.6843e-006 |
| 2 | 1.26355207730157e-001 | 1.26566988159545e-001 | 1.6761e-003 | 0.4651e-006 |
| 3 | 9.91977759128186e-002 | 9.93925089271431e-002 | 1.9631e-003 | 0.2693e-006 |
| 4 | 7.52968627216139e-002 | 7.53482005051614e-002 | 6.8181e-004 | 0.1415e-006 |
| 5 | 5.50992897250076e-002 | 5.50937306915325e-002 | 1.0089e-004 | 0.0613e-006 |
| 6 | 3.87720340552233e-002 | 3.87740291567117e-002 | 5.1457e-005 | 0.0225e-006 |
| 7 | 2.61864259662364e-002 | 2.61965768564899e-002 | 3.8764e-004 | 0.0061e-006 |
| 8 | 1.69583460230047e-002 | 1.69644786561981e-002 | 3.6163e-004 | 0.0014e-006 |
| 9 | 1.05302881042214e-002 | 1.05300540407586e-002 | 2.2228e-005 | 0.0004e-006 |
| 10 | 6.27608178211405e-003 | 6.28016646975848e-003 | 6.5083e-004 | 0.0003e-006 |
| 11 | 3.59742492542161e-003 | 3.61024314711437e-003 | 3.5632e-003 | 0.0002e-006 |
| 12 | 1.98859806602346e-003 | 1.99720241933600e-003 | 4.3268e-003 | 0.0001e-006 |
| 13 | 1.06361718551436e-003 | 1.06523254224819e-003 | 1.5187e-003 | 0.0001e-006 |
| 14 | 5.52535380835848e-004 | 5.55738846100656e-004 | 5.7978e-003 | 3.7952e-012 |
| 15 | 2.79804060541021e-004 | 2.79858187952667e-004 | 1.9345e-004 | 1.7216e-012 |
| 16 | 1.38716564985509e-004 | 1.38766753067992e-004 | 3.6180e-004 | 7.5632e-013 |
| 17 | 6.74617452427104e-005 | 6.73796527812621e-005 | 1.2169e-003 | 3.1713e-013 |

TABLE 4 $m$=5, $N$=50,000

| $\gamma_{av}$ (dB) | Quadrature | Monte Carlo | Relative Error | Estimator Variance |
|---|---|---|---|---|
| 0 | 1.80862198964748e-001 | 1.81077403546023e-001 | 1.1899e-003 | 0.2232e-006 |
| 1 | 1.48692399384738e-001 | 1.48459112083375e-001 | 1.5689e-003 | 0.1760e-006 |
| 2 | 1.18484239649771e-001 | 1.18492405076103e-001 | 6.8916e-005 | 0.1273e-006 |
| 3 | 9.11198182344937e-002 | 9.14565833460168e-002 | 3.6958e-003 | 0.0771e-006 |
| 4 | 6.73140277752907e-002 | 6.77445567402741e-002 | 6.3958e-003 | 0.0400e-006 |
| 5 | 4.75334135754589e-002 | 4.73720955500846e-002 | 3.3938e-003 | 0.0188e-006 |
| 6 | 3.19281433175661e-002 | 3.19455540935450e-002 | 5.4531e-004 | 0.0071e-006 |
| 7 | 2.03094550907598e-002 | 2.02164900899754e-002 | 4.5774e-003 | 0.0022e-006 |
| 8 | 1.21896874418357e-002 | 1.21894585147488e-002 | 1.8780e-005 | 0.0006e-006 |
| 9 | 6.88633412922880e-003 | 6.88699378056795e-003 | 9.5791e-005 | 0.0001e-006 |
| 10 | 3.65797504636271e-003 | 3.65626179310632e-003 | 4.6836e-004 | 1.5451e-011 |
| 11 | 1.82824824354838e-003 | 1.82670613552302e-003 | 8.4349e-004 | 3.2726e-012 |
| 12 | 8.61296319495424e-004 | 8.61171674903717e-004 | 1.4472e-004 | 1.6597e-012 |
| 13 | 3.84024214297071e-004 | 3.82595340605489e-004 | 3.7208e-003 | 8.8893e-013 |
| 14 | 1.62580898382279e-004 | 1.62527523404886e-004 | 3.2829e-004 | 3.4732e-013 |
| 15 | 6.58910400214678e-005 | 6.56037294831911e-005 | 4.3604e-003 | 1.1989e-013 |
| 16 | 2.55904610614731e-005 | 2.54351649812293e-005 | 6.0685e-003 | 3.4788e-014 |
| 17 | 9.58853443650534e-006 | 9.69325861107578e-006 | 1.0922e-002 | 1.0710e-014 |
| 18 | 3.48504249549011e-006 | 3.45875637643274e-006 | 7.5426e-003 | 3.5108e-015 |
| 19 | 1.23470250592159e-006 | 1.25664822568459e-006 | 1.7774e-002 | 6.3846e-016 |
| 20 | 4.28199891953809e-007 | 4.43127893087176e-007 | 3.4862e-002 | 2.2348e-016 |

TABLE 5 $m$=10, $N$=50,000

| $\gamma_{av}$ (dB) | Quadrature | Monte Carlo | Relative Error | Estimator Variance |
|---|---|---|---|---|
| 0 | 1.72603300437725e-001 | 1.73481435910224e-001 | 5.0876e-003 | 0.2865e-006 |
| 1 | 1.39684206914816e-001 | 1.39319194200152e-001 | 2.6131e-003 | 0.2471e-006 |
| 2 | 1.09012194839582e-001 | 1.09370449721748e-001 | 3.2864e-003 | 0.1738e-006 |
| 3 | 8.15239122311026e-002 | 8.12984819815417e-002 | 2.7652e-003 | 0.1128e-006 |
| 4 | 5.79766320128721e-002 | 5.81844694459776e-002 | 3.5848e-003 | 0.0650e-006 |
| 5 | 3.88602975244914e-002 | 3.85513874788571e-002 | 7.9492e-003 | 0.0308e-006 |
| 6 | 2.43053472639699e-002 | 2.40908930966324e-002 | 8.8233e-003 | 0.0135e-006 |
| 7 | 1.40338681316313e-002 | 1.39210669120146e-002 | 8.0378e-003 | 0.0042e-006 |
| 8 | 7.39882870625043e-003 | 7.48739810919140e-003 | 1.1971e-002 | 0.0011e-006 |
| 9 | 3.52391636615921e-003 | 3.51765960942736e-003 | 1.7755e-003 | 0.0002e-006 |
| 10 | 1.50162026497609e-003 | 1.50480536458026e-003 | 2.1211e-003 | 2.6573e-011 |
| 11 | 5.68487203423154e-004 | 5.68702126642903e-004 | 3.7806e-004 | 2.4512e-012 |
| 12 | 1.89833010412242e-004 | 1.89586799998451e-004 | 1.2969e-003 | 1.3951e-013 |
| 13 | 5.60270328291789e-005 | 5.57730699385937e-005 | 4.5328e-003 | 4.9360e-015 |
| 14 | 1.45456045354261e-005 | 1.44357571106622e-005 | 7.5519e-003 | 1.5942e-016 |
| 15 | 3.33646389596764e-006 | 3.31728963534209e-006 | 5.7469e-003 | 3.7849e-017 |
| 16 | 6.81761241701623e-007 | 6.75333729340419e-007 | 9.4278e-003 | 4.7275e-018 |
| 17 | 1.25221203905896e-007 | 1.25998503494660e-007 | 6.2074e-003 | 4.5175e-019 |
| 18 | 2.08915909571968e-008 | 2.08322432925173e-008 | 2.8407e-003 | 3.0701e-020 |
| 19 | 3.20154678503786e-009 | 3.13669764399620e-009 | 2.0256e-002 | 2.5365e-021 |
| 20 | 4.55720769286559e-010 | 4.38437151565145e-010 | 3.7926e-002 | 1.3061e-022 |

The last two sets of simulations are for the case where $m$=5 (Table 4) and $m$=10 (Table 5). In both cases, $N$=50,000. It is seen that the relative errors behave in much the same way as in previous simulations.

In all the simulations considered, it has been found that increasing $N$ brought an immediate improvement in the performance of (10) for smaller $\gamma_{av}$ but as $\gamma_{av}$ increased, the performance reduced.

Finally, it is noted that the results generated through the estimator (10) are substantial improvements on the corresponding estimates in (Weinberg 2008), and the estimator emulates the performance of the AWGN efficient estimator (10) in the same paper.

## Conclusion

By means of the Poisson-Marcum Q-Function association from (Weinberg 2006), a general Monte Carlo estimator has been derived, enabling the estimation of DQPSK BERs under Nakagami fading channels. This estimator uses knowledge of conditional distributions associated with the Poisson-Marcum Q-Function relationship. In the context of a simple example, it is demonstrated mathematically through simulations that the new estimator has superior performance to that proposed in (Weinberg 2008). In particular, the new estimator is free from simulation difficulties due to sampling from a problematic joint distribution and yielding more consistent results. It was found that BERs with small $\gamma_{av}$ could be estimated for sample sizes of order $N$ = 50,000, but for larger $\gamma_{av}$ a much greater sample size is required. These simulation results are consistent with those obtained through numerical integration.

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